Math 565: Functional Analysis
Lecture 24

In fact, the last class of examples are all Hilbert-Schmidt operators on $L^2(X,\mu)$, and the proof is left as HW.

Theorem. Let (X,μ) be a σ-finite measure space. Then a bdd op. T on $L^2(X,\mu)$ is Hilbert-Schmidt $\iff T = T_K$ for some $K \in L^2(X\times X, \mu\times\mu)$.

Spectral theory of compact operators.

Throughout, let H denote an arbitrary Hilbert space.

Def. A bdd operator $T \in B(H)$ is called *self-adjoint* if $T = T^*$.

Examples. (a) Orthogonal projections on closed subspaces of H are self-adjoint.

(b) If T is self-adjoint and $p \in \mathbb{R}[t]$ then $p(T)$ is self-adjoint.

(c) Recall that if $\{e_i\}_{i\in I}$ is an ON basis for H and $\lambda_i \to 0$ for some $(\lambda_i)_{i\in I} \in \ell^\infty(I)$, then the operator $T : H \to H$ defined by $Te_i := \lambda_i e_i$ is compact. Its adjoint is given by $T^* e_i := \overline{\lambda_i} e_i$, which is also compact. Thus, T is self-adjoint $\iff$ $\lambda_i \in \mathbb{R}$ $\forall i \in I$.

(d) For a measure space (X,μ) and $\varphi \in L^\infty(X,\mu)$, we define the *multiplication operator* $M_\varphi : L^2(X,\mu) \to L^2(X,\mu)$ associated to φ by $M_\varphi f := \varphi \cdot f$. Then $\|M_\varphi\| = \|\varphi\|_\infty$ and $M_\varphi^* = M_{\overline{\varphi}}$. Thus, M_φ is self-adjoint $\iff$ φ is real-valued.

Def. For a Banach space X and $T \in B(X)$, the *point spectrum* of T is the set
$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq 0\},$$
i.e. $\sigma_p(T)$ is the set of eigenvalues. For $\lambda \in \sigma_p(T)$, its *eigenspace* is $E_\lambda := \ker(T - \lambda I)$.

Def. Let $\{H_i\}_{i\in I}$ be a collection of pairwise orthogonal subspaces of H. The direct sum of these subspaces is the subspace:

$$\bigoplus_{i\in I} H_i := \{\sum_{i\in I} x_i : x_i \in H_i \ \forall i\in I \text{ and } \sum_{i\in I} x_i \text{ converges in } H\} = \overline{\bigcup_{\substack{I_0\subseteq I\\ \text{finite}}} \bigoplus_{i\in I_0} H_i}.$$

Then the analog of Parseval's identity holds: for each $x\in \bigoplus_{i\in I} H_i$, $\|x\|^2 = \sum_{i\in I} \|\mathrm{proj}_{H_i} x\|^2$.

Spectral theorem for compact self-adjoint operators. Every compact self-adjoint $T\in B(H)$ admits an ON basis $\{e_i\}_{i\in I}$ of eigenvectors; equivalently,

$$H = \bigoplus_{\lambda\in\sigma_p(T)} E_\lambda.$$

Furthermore, $\dim(E_\lambda) < \infty$ and $\{\lambda\in\sigma_p(T) : |\lambda|\geq \varepsilon\}$ is finite $\forall \varepsilon>0$.

To prove this, we need to record some properties of self-adjoint operators. Observe that for any linear $T: H\to H$, we have

$$\|T\| = \sup_{\|x\|,\|y\|=1} |\langle Tx, y\rangle|. \qquad (*)$$

Indeed, $\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle Tx, y\rangle|$.

Properties of self-adjoint operators. Let $T\in B(H)$ be self-adjoint. Then:

(a) $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

(b) If $W\subseteq H$ is T-invariant (i.e. $T(W)\subseteq W$) then $W^\perp$ is also T-invariant.

(c) If $W\subseteq H$ is T-invariant and closed then $(T|_W)^* = (T^*|_W)$, hence $T|_W$ is self-adjoint.

(d) $\langle Tx, x\rangle\in\mathbb{R}$ $\forall x\in H$. In particular, $\sigma_p(T)\subseteq\mathbb{R}$.

(e) If $\lambda_0\neq\lambda_1$ are in $\sigma_p(T)$, then $E_{\lambda_0}\perp E_{\lambda_1}$.

Proof. (b) If $w\in W$ and $w_\perp\in W^\perp$, then $\langle w, Tw_\perp\rangle = \langle w, T^* w_\perp\rangle = \langle Tw, w_\perp\rangle = 0$ hence

$Tw \in W$, hence $Tw_2 \in W^\perp$.

(c) Observe that $T^*|_W$ satisfies the adjoint identity, so $(T|_W)^* = (T^*|_W) = T|_W$.

(d) $\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$, so $\langle Tx, x\rangle \in \mathbb{R}$. If $Tx = \lambda x$ for some $0 \neq x \in H$, then $\lambda \|x\|^2 = \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle \in \mathbb{R}$, so $\lambda \in \mathbb{R}$.

(e) Let $x_i \in E_{\lambda_i}$, $i = 0,1$. Then $\lambda_0\langle x_0, x_1\rangle = \langle \lambda_0 x_0, x_1\rangle = \langle Tx_0, x_1\rangle = \langle x_0, Tx_1\rangle = \langle x_0, \lambda_1 x_1\rangle =$
$= \lambda_1\langle x_0, x_1\rangle$, so $(\lambda_0 - \lambda_1)\langle x_0, x_1\rangle = 0$ hence $\langle x_0, x_1\rangle = 0$.

(a) By $(*)$ above, $\alpha := \sup_{\|x\|=1} |\langle Tx, x\rangle| \le \|T\|$ and we show that $|\langle Tx, y\rangle| \le \alpha$ $\forall \|x\|, \|y\| \le 1$. Fix $x, y \in H$ with $\|x\|, \|y\| = 1$. By multiplying x with a modulus 1 complex number, we may assume $\langle Tx, y\rangle \geq 0$. We aim to express $\langle Tx, y\rangle$ via terms $\langle Tv, v\rangle$.

(i) $\langle T(x+y), x+y\rangle = \langle Tx, x\rangle + \langle Ty, y\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle = \langle Tx, x\rangle + \langle Ty, y\rangle + 2\langle Tx, y\rangle$

(ii) $\langle T(x-y), x-y\rangle = \langle Tx, x\rangle + \langle Ty, y\rangle - 2\langle Tx, y\rangle$.

Subtracting (i)−(ii), we get

$$4\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \le \alpha\left(\|x+y\|^2 + \|x-y\|^2\right)$$
$$= \alpha\left(2\|x\|^2 + 2\|y\|^2\right) = 4\alpha,$$

hence $\langle Tx, y\rangle \le \alpha$. □

Obs. For any Banach space X and $T \in B(X)$, if $\lambda \in \sigma_p(T)$ then $|\lambda| \le \|T\|$.

Proof. If $x \in E_\lambda$ then $\|Tx\| = \|\lambda x\| = |\lambda|\,\|x\|$, so $|\lambda| \le \|T\|$. □

Main lemma. Every compact self-adjoint operator $T \in B(H)$ has an eigenvalue $\lambda = \pm\|T\|$.

Proof. Since $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$, $\exists\, (x_n) \subseteq H$ of unit vectors s.t. $\lim_n |\langle Tx_n, x_n\rangle| = \|T\|$. By switching to a subsequence, we may assume that $\lim_n \langle Tx_n, x_n\rangle = \lambda$ and $|\lambda| = \|T\|$. By the compactness of T, (Tx_n) has a convergent subsequence, so switching to it, we may assume that $\lim_n Tx_n = y \in H$. May assume $T \neq 0$, so $\lambda = \pm\|T\| \neq 0$. We verify that $x := \lambda^{-1} y$ is an eigenvector. Note:

$\|Tx_n - \lambda x_n\|^2 = \langle Tx_n - \lambda x_n, Tx_n - \lambda x_n\rangle = \|Tx_n\|^2 + |\lambda|^2\|x_n\|^2 - 2\lambda\langle Tx_n, x_n\rangle$ because $\langle Tx_n, x_n\rangle \in \mathbb{R}$ by self-adjointness of T. Thus,

$$\|Tx_n - \lambda x_n\|^2 \le \|T\|^2 + \|T\|^2 - 2\lambda\langle Tx_n, x_n\rangle \xrightarrow[n\to\infty]{} 2\|T\|^2 - 2\lambda^2 = 0,$$ hence $\lambda x_n \to y$, i.e.

$x_n \to x$. By the continuity of T, then $Tx_n \to Tx$ but also $Tx_n \to y = \lambda x$, so $Tx = \lambda x$. Also $x \neq 0$ because $\langle Tx_n, x_n \rangle \to \lambda \neq 0$ and also $\lambda = \langle Tx, x \rangle$. $\square$

Proof of spectral thm for compact self-adjoint operators. By Zorn's lemma, let $\{e_i\}_{i\in I}$ be a maximal ON family of eigenvectors of T. Let $W := \overline{\operatorname{span}\{e_i\}_{i\in I}}$.
Note that W is T-invariant hence $W^\perp$ is T-invariant and $T|_{W^\perp}$ is still self-adjoint, so if $W^\perp \neq 0$ then $T|_{W^\perp}$ would have an eigenvector (with eigenvalue $\lambda = \pm \|T|_{W^\perp}\|$), contradicting the maximality of $\{e_i\}_{i\in I}$. $\square$

Def. An operator $T \in B(H)$ is called normal if $TT^* = T^*T$.

Example. (a) Self-adjoint operators are normal.
(b) If T is normal and $p \in \mathbb{C}[t]$, then $p(T)$ is normal.
(c) If T is unitary, then T is normal because $T^* = T^{-1}$.

Decomposition into self-adjoints. For any $T \in B(H)$ there are unique self-adjoint $T_1, T_2 \in B(H)$ with $T = T_1 + iT_2$; in fact, $T_1 := \frac{1}{2}(T + T^*)$ and $T_2 := \frac{1}{2i}(T - T^*)$.
In particular, T is normal $\iff$ T_1 and T_2 commute.

Proof. That T_1, T_2 as given in the statement are self-adjoint and $T = T_1 + iT_2$ is apparent, and the characterization of normality is a straightforward verification. To prove the uniqueness of such a decomposition, suppose $T = T_1' + iT_2'$ is another such decomposition. Then

$$0 = T - T = T_1 - T_1' + i(T_2 - T_2')$$
$$0 = T^* - T^* = T_1 - T_1' - i(T_2 - T_2')$$

and adding these two equalities gives $T_1 = T_1'$, while subtracting gives $T_2 = T_2'$. $\square$